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Confining Phase of N=1 Supersymmetric Gauge Theories and N=2 Massless Solitons

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Abstract

Effective superpotentials for the phase with a confined photon are obtained in $N = 1$ supersymmetric gauge theories. We use the results to derive the hyperelliptic curves which describe the Coulomb phase of $N = 2$ theories with classical gauge groups, and thus extending the prior result for $SU(N_c)$ gauge theory by Elitzur et al. Moreover, adjusting the coupling constants in $N = 1$ effective superpotentials to the values of $N = 2$ non-trivial critical points we find new classes of $N = 1$ superconformal field theories with an adjoint matter with a superpotential.

Exact descriptions of strong-coupling dynamics of supersymmetric gauge theories in four dimensions have been obtained on the basis of the idea of duality and holomorphy [1], [2], [3], [4]. In these exact solutions an important feature in common is that singularities of quantum moduli space of the theory correspond to the appearance of massless solitons. Near the singularity, therefore, we observe interesting non-perturbative properties of supersymmetric gauge theories.

In an $N = 2$ case the Coulomb phase admits a beautiful mathematical description according to which massless solitons are recognized as vanishing cycles associated with the degeneracy of hyperelliptic curves. In order to explore physics near $N = 2$ singularities the microscopic superpotential explicitly breaking $N = 2$ to $N = 1$ supersymmetry is often considered [2], [3], [5], [6], [7]. Examining the resulting superpotential for a low-energy effective Abelian theory it is found that the generic $N = 2$ vacuum is lifted and only the singular loci of moduli space remain as the $N = 1$ vacua where monopoles or dyons can condense.

Alternatively we may start with a microscopic $N = 1$ theory which we introduce by perturbing an $N = 2$ theory by adding a tree-level superpotential built out of the Casimirs of the adjoint field in the vector multiplet [5], [8], [9]. Let us concentrate on a phase with a single confined photon in our $N = 1$ theory which corresponds to the classical $SU(2) \times U(1)^{r-1}$ vacua with r being the rank of the gauge group. Then the low-energy effective theory containing non-perturbative effect provides us with the data of the vacua with massless solitons [10], [5]. This will enable us to reconstruct hyperelliptic curves for $N = 2$ theories.

Along this line of thought Elitzur et al. [11] have recently obtained $N = 1$ effective superpotentials from which the curves for the Coulomb phase of $N = 2$ $SU(N_c)$ gauge theories can be verified. Our purpose in this paper is to extend their result to the case of arbitrary classical gauge group. In what follows we first derive $N = 1$ effective superpotentials which can be used to reproduce the $N = 2$ curves for classical gauge groups. We next discuss $N = 1$ superconformal field theories based on our results for superpotentials.

We begin with briefly reviewing the results of [11] on $SU(N_c)$ gauge theories. The gauge symmetry breaks down to $U(1)^{N_c-1}$ in the Coulomb phase of $N = 2$ $SU(N_c)$ Yang-Mills theories. Near the singularity of a single massless dyon we have a photon coupled

to the light dyon hypermultiplet while the photons for the rest $U(1)^{N_c-2}$ factors remain free. We now perturb the theory by adding a tree-level superpotential

$$W=\sum_{n=1}^{N_c} g_n u_n, \qquad u_n=\frac{1}{n}\mathrm{Tr}\,\Phi^n, \tag{1}$$

where Φ is the adjoint $N=1$ superfield in the $N=2$ vectormultiplet and g_1 is an auxiliary field implementing $\mathrm{Tr}\,\Phi=0$. In view of the macroscopic theory, we see that under the perturbation by (1) only the $N=2$ singular loci survive as the $N=1$ vacua where a single photon is confined and the $U(1)^{N_c-2}$ factors decouple.

The result should be directly recovered when we start with the microscopic $N=1$ $SU(N_c)$ gauge theory which is obtained from $N=2$ $SU(N_c)$ Yang-Mills theory perturbed by (1). For this we study the vacuum with unbroken $SU(2)\times U(1)^{N_c-2}$. The classical vacua of the theory are determined by the equation of motion $W'(\Phi)=\sum_{i=1}^{N_c} g_i\Phi^{i-1}=0$. Then the roots a_i of

$$W'(x)=\sum_{i=1}^{N_c} g_i x^{i-1}=g_{N_c}\prod_{i=1}^{N_c-1}(x-a_i) \tag{2}$$

give the eigenvalues of Φ. In particular the unbroken $SU(2)\times U(1)^{N_c-2}$ vacuum is described by

$$\Phi=\mathrm{diag}(a_1,a_1,a_2,a_3,\cdots,a_{N_c-1}). \tag{3}$$

In the low-energy limit the adjoint superfield for $SU(2)$ becomes massive and will be decoupled. We are then left with an $N=1$ $SU(2)$ Yang-Mills theory which is in the confining phase and the photon multiplets for $U(1)^{N_c-2}$ are decoupled.

The relation between the high-energy $SU(N_c)$ scale Λ and the low-energy $SU(2)$ scale Λ_L is determined by first matching at the scale of $SU(N_c)/SU(2)$ W bosons and then by matching at the $SU(2)$ adjoint mass $M_{\rm ad}$. One finds [12], [11]

$$\Lambda^{2N_c}={\Lambda_L}^{3\cdot 2}\left(\prod_{i=2}^{N_c-1}(a_1-a_i)\right)^2 (M_{\rm ad})^{-2}. \tag{4}$$

To compute $M_{\rm ad}$ we decompose

$$\Phi=\Phi_{cl}+\delta\Phi+\delta\tilde{\Phi}, \tag{5}$$

where $\delta\Phi$ denotes the fluctuation along the unbroken $SU(2)$ direction and $\delta\tilde{\Phi}$ along the other directions. Substituting this into W we have

$$
\begin{aligned}
W &= W_{cl} + \sum_{i=2}^{N_c} g_i \frac{i-1}{2} \operatorname{Tr}(\delta\Phi^2 \Phi_{cl}^{i-2}) + \cdots \\
&= W_{cl} + \frac{1}{2} W''(a_1) \operatorname{Tr}\delta\Phi^2 + \cdots \\
&= W_{cl} + \frac{1}{2}\, g_{N_c} \prod_{i=2}^{N_c} (a_1 - a_i) \operatorname{Tr}\delta\Phi^2 + \cdots, \qquad (6)
\end{aligned}
$$

where $[\delta\Phi, \Phi_{cl}] = 0$ has been used and W_{cl} is the tree-level superpotential evaluated in the classical vacuum. Hence, $M_{\rm ad} = g_{N_c} \prod_{i=2}^{N_c-1}(a_1 - a_i)$ and the relation (4) reduces to

$$
{\Lambda_L}^6 = g_{N_c}^2 \Lambda^{2N_c}. \qquad (7)
$$

Since the gaugino condensation dynamically generates the superpotential in the $N = 1$ $SU(2)$ theory the low-energy effective superpotential finally takes the form [11]

$$
W_L = W_{cl} \pm 2{\Lambda_L}^3 = W_{cl} \pm 2 g_{N_c} \Lambda^{N_c}. \qquad (8)
$$

We simply assume here that the superpotential (8) is exact for any values of the parameters. (This is equivalent to assume $W_\Delta = 0$ [10], [11].) From (8) we obtain

$$
\langle u_n \rangle = \frac{\partial W_L}{\partial g_n} = u_n^{cl}(g) \pm 2\Lambda^{N_c} \delta_{n,N_c} \qquad (9)
$$

with u_n^{cl} being a classical value of u_n. As we argued above these vacua should correspond to the singular loci of $N = 2$ massless dyons. This can be easily confirmed by plugging (9) in the $N = 2$ $SU(N_c)$ curve [13], [14]

$$
y^2 = \langle \det(x - \Phi) \rangle^2 - 4\Lambda^{2N_c} = \left(x^{N_c} - \sum_{i=2}^{N_c} \langle s_i \rangle x^{N_c - i} \right)^2 - 4\Lambda^{2N_c}, \qquad (10)
$$

where

$$
k s_k + \sum_{i=1}^{k} i s_{k-i} u_i = 0, \qquad k = 1, 2, \cdots \qquad (11)
$$

with $s_0 = -1$ and $s_1 = u_1 = 0$. We have

$$
\begin{aligned}
y^2 &= \left(x^{N_c} - s_2^{cl} x^{N_c-2} - \cdots - s_{N_c}^{cl} \right) \left(x^{N_c} - s_2^{cl} x^{N_c-2} - \cdots - s_{N_c}^{cl} \pm 4\Lambda^{N_c} \right) \\
&= (x - a_1)^2 (x - a_2) \cdots (x - a_{N_c-1}) \left((x - a_1)^2 (x - a_2) \cdots (x - a_{N_c-1}) \pm 4\Lambda^{N_c} \right) (12)
\end{aligned}
$$

Since the curve exhibits the quadratic degeneracy we are exactly at the singular point of a massless dyon in the $N = 2$ $SU(N_c)$ Yang-Mills vacuum.

Let us now apply our procedure to the $N = 2$ $SO(2N_c)$ Yang-Mills theory. We take a tree-level superpotential to break $N = 2$ to $N = 1$ as

$$W = \sum_{n=1}^{N_c-1} g_{2n} u_{2n} + \lambda v, \tag{13}$$

where

$$\begin{aligned} u_{2n} &= \frac{1}{2n} \mathrm{Tr}\, \Phi^{2n}, \\ v = \mathrm{Pf}\, \Phi &= \frac{1}{2^{N_c} N_c!} \epsilon_{i_1 i_2 j_1 j_2 \cdots} \Phi^{i_1 i_2} \Phi^{j_1 j_2} \cdots \end{aligned} \tag{14}$$

and the adjoint superfield Φ is an antisymmetric $2N_c \times 2N_c$ tensor. This theory has classical vacua which satisfy the condition

$$W'(\Phi) = \sum_{i=1}^{N_c-1} g_{2i} (\Phi^{2i-1})_{ij} - \frac{\lambda}{2^{N_c}(N_c-1)!} \epsilon_{\,i\,j\,k_1 k_2 l_1 l_2 \cdots} \Phi^{k_1 k_2} \Phi^{l_1 l_2} \cdots = 0. \tag{15}$$

For the skew-diagonal form of Φ

$$\Phi = \mathrm{diag}(\sigma_2 e_0,\ \sigma_2 e_1,\ \sigma_2 e_2, \cdots, \sigma_2 e_{N_c-1}), \qquad \sigma_2 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{16}$$

the vacuum condition (15) becomes

$$\sum_{i=1}^{N_c-1} g_{2i} (-1)^{i-1} {e_n}^{2i-1} + (-i)^{N_c} \frac{\lambda}{2e_n} \prod_{i=0}^{N_c-1} e_i = 0\,, \qquad 0 \le n \le N_c - 1. \tag{17}$$

Thus we see that e_n ($\neq 0$) are the roots of $f(x)$ defined by

$$f(x) = \sum_{i=1}^{N_c-1} g_{2i} x^{2i} + d, \tag{18}$$

where we put $d = (-i)^{N_c} \frac{1}{2} \lambda \prod_{i=0}^{N_c-1} e_i$.

Since our main concern is the vacuum with a single confined photon we focus on the unbroken $SU(2) \times U(1)^{N_c-1}$ vacuum. Thus writing (18) as

$$f(x) = g_{2(N_c-1)} \prod_{i=1}^{N_c-1} (x^2 - a_i^2), \tag{19}$$

we take

$$\Phi = \mathrm{diag}(\sigma_2 a_1,\ \sigma_2 a_1,\ \sigma_2 a_2, \cdots, \sigma_2 a_{N_c-1}) \tag{20}$$

with $d = (-i)^{N_c}\frac{1}{2}\lambda a_1^2 \prod_{i=2}^{N_c-1} a_i$. We then make the scale matching between the high-energy $SO(2N_c)$ scale Λ and the low-energy $SU(2)$ scale Λ_L. Following the steps as in the $SU(N_c)$ case yields

$$\Lambda^{2\cdot 2(N_c-1)} = {\Lambda_L}^{3\cdot 2}\left(\prod_{i=2}^{N_c-1}(a_1^2 - a_i^2)\right)^2 (M_{\mathrm{ad}})^{-2}, \tag{21}$$

where the factor arising through the Higgs mechanism is easily calculated in an explicit basis of $SO(2N_c)$. In order to evaluate the $SU(2)$ adjoint mass M_{ad} we first substitute the decomposition (5) in W and proceed as follows:

$$\begin{aligned} W &= W_{cl} + \sum_{i=1}^{N_c-1} g_i \frac{2i-1}{2}\,\mathrm{Tr}\,(\delta\Phi^2\Phi_{cl}^{2i-2}) + \lambda\,(\mathrm{Pf}_4\delta\Phi)\left(\mathrm{Pf}_{2(N_c-2)}\Phi_{cl}\right) + \cdots \\ &= W_{cl} + \sum_{i=1}^{N_c-1} g_i \frac{2i-1}{2}\,\mathrm{Tr}\,(\delta\Phi^2\Phi_{cl}^{2i-2}) + \lambda\left(\frac{1}{4}\mathrm{Tr}\,\delta\Phi^2\right)\left(\prod_{k=2}^{N_c-1}(-ia_k)\right) + \cdots \\ &= W_{cl} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{x}\right)\bigg|_{x=a_1} \mathrm{Tr}\,\delta\Phi^2 + \cdots \\ &= W_{cl} + g_{2(N_c-1)}\prod_{i=2}^{N_c-1}(a_1^2 - a_i^2)\,\mathrm{Tr}\,\delta\Phi^2 + \cdots, \end{aligned} \tag{22}$$

where Pf_4 is the Pfaffian of a upper-left 4×4 sub-matrix and $\mathrm{Pf}_{2(N_c-2)}$ is the Pfaffian of a lower-right $2(N_c-2)\times 2(N_c-2)$ sub-matrix. Thus we observe that M_{ad} cancels out the Higgs factor in (21), which leads to ${\Lambda_L}^6 = g_{2(N_c-1)}^2 \Lambda^{4(N_c-1)}$. The low-energy superpotential is now given by

$$W_L = W_{cl} \pm 2{\Lambda_L}^3 = W_{cl} \pm 2 g_{2(N_c-1)}\Lambda^{2(N_c-1)}, \tag{23}$$

where the second term is due to the gaugino condensation in the low-energy $SU(2)$ theory.

The vacuum expectation values of gauge invariants are obtained from W_L as

$$\begin{aligned} \langle u_{2n}\rangle &= \frac{\partial W_L}{\partial g_{2n}} = u_{2n}^{cl}(g,\lambda) \pm 2\Lambda^{2(N_c-1)}\delta_{n,N_c-1}, \\ \langle v\rangle &= \frac{\partial W_L}{\partial \lambda} = v_{cl}(g,\lambda). \end{aligned} \tag{24}$$

The curve for $N=2$ $SO(2N_c)$ is known to be [15]

$$y^2 = \langle \det(x-\Phi)\rangle^2 - 4\Lambda^{4(N_c-1)}x^4$$

$$= \left(x^{2N_c} - \sum_{i=1}^{N_c-1} \langle s_{2i}\rangle x^{2(N_c-i)} + \langle v\rangle^2\right)^2 - 4\Lambda^{4(N_c-1)}x^4, \tag{25}$$

where

$$ks_k + \sum_{i=1}^{k} i s_{k-i} u_{2i} = 0, \qquad k = 1, 2, \cdots \tag{26}$$

with $s_0 = -1$. At the values (24) of the moduli coordinates we see the quadratic degeneracy

$$\begin{aligned} y^2 &= \left(x^{2N_c} - s_2^{cl} x^{2(N_c-1)} - \cdots - s_{2(N_c-1)}^{cl} x^2 + v_{cl}^2\right) \\ &\quad \times \left(x^{2N_c} - s_2^{cl} x^{2(N_c-1)} - \cdots - s_{2(N_c-1)}^{cl} x^2 + v_{cl}^2 \pm 4\Lambda^{2(N_c-1)}x^2\right) \\ &= (x^2 - a_1^2)^2 (x^2 - a_2^2) \cdots (x^2 - a_{N_c-1}^2) \\ &\quad \times \Big((x^2 - a_1^2)^2 (x^2 - a_2^2) \cdots (x^2 - a_{N_c-1}^2) \pm 4\Lambda^{2(N_c-1)}x^2\Big). \end{aligned} \tag{27}$$

This is our desired result. Notice that the apparent singularity at $\langle v\rangle = 0$ is not realized in our $N = 1$ theory. Thus the point $\langle v\rangle = 0$ does not correspond to massless solitons in agreement with the result of [15].

Our next task is to study the $SO(2N_c + 1)$ gauge theory. A tree-level superpotential breaking $N = 2$ to $N = 1$ is assumed to be

$$W = \sum_{n=1}^{N_c} g_{2n} u_{2n}, \qquad u_{2n} = \frac{1}{2n} \mathrm{Tr}\, \Phi^{2n}. \tag{28}$$

The classical vacua obey $W'(\Phi) = \sum_{i=1}^{N_c} g_{2i} \Phi^{2i-1} = 0$. The eigenvalues of Φ are given by the roots a_i of

$$W'(x) = \sum_{i=1}^{N_c} g_{2i} x^{2i-1} = g_{2N_c} x \prod_{i=1}^{N_c-1} (x^2 - a_i^2). \tag{29}$$

As in the previous consideration we take the $SU(2) \times U(1)^{N_c-1}$ vacuum. Notice that there are two ways of breaking $SO(2N_c + 1)$ to $SU(2) \times U(1)^{N_c-1}$. One is to take all the eigenvalues distinct (corresponding to $SO(3) \times U(1)^{N_c-1}$). The other is to choose two eigenvalues coinciding and the rest distinct (corresponding to $SU(2) \times U(1)^{N_c-1}$ with $a_i \neq 0$). Here we examine the latter case

$$\Phi = \mathrm{diag}(\sigma_2 a_1,\ \sigma_2 a_1,\ \sigma_2 a_2, \cdots, \sigma_2 a_{N_c-1},\ 0), \qquad \sigma_2 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{30}$$

In this vacuum the high-energy $SO(2N_c+1)$ scale Λ and the low-energy $SU(2)$ scale Λ_L are related by

$$\Lambda^{2\cdot(2N_c-1)} = {\Lambda_L}^{3\cdot 2} a_1^2 \left(\prod_{i=2}^{N_c-1} (a_1^2 - a_i^2) \right)^2 (M_{\rm ad})^{-2}, \tag{31}$$

where the $SU(2)$ adjoint mass $M_{\rm ad}$ is read off from

$$\begin{aligned} W &= W_{cl} + \sum_{i=1}^{N_c} g_{2i} \frac{2i-1}{2} \,\mathrm{Tr}\,(\delta\Phi^2 \Phi_{cl}^{2i-2}) + \cdots \\ &= W_{cl} + \frac{1}{2} W''(a_1) \,\mathrm{Tr}\,\delta\Phi^2 + \cdots \\ &= W_{cl} + g_{2N_c} a_1^2 \prod_{i=2}^{N_c-1} (a_1^2 - a_i^2) \,\mathrm{Tr}\,\delta\Phi^2 + \cdots . \end{aligned} \tag{32}$$

So, we obtain ${\Lambda_L}^6 = g_{2N_c}^2 a_1^2 \Lambda^{2(2N_c-1)}$. The low-energy effective superpotential becomes

$$W_L = W_{cl} \pm 2{\Lambda_L}^3 = W_{cl} \pm 2 g_{2N_c} a_1 \Lambda^{2N_c-1}. \tag{33}$$

If we assume $W_\Delta = 0$ the expectation values $\langle u_{2i} \rangle$ are calculated from W_L by expressing a_1 as a function of g_{2i}.

For the sake of illustration let us discuss the $SO(5)$ theory explicitly. From (33) we get

$$\begin{aligned} \langle u_2 \rangle &= 2a_1^2 \pm \frac{1}{a_1} \Lambda^3, \\ \langle u_4 \rangle &= a_1^4 \mp a_1 \Lambda^3 \end{aligned} \tag{34}$$

and $a_1^2 = -g_2/g_4$. We eliminate a_1 from (34) to obtain

$$27\Lambda^{12} - \Lambda^6 u_2^3 + 36\Lambda^6 u_2 u_4 - u_2^4 u_4 + 8u_2^2 u_4^2 - 16u_4^3 = 0. \tag{35}$$

This should be compared with the $N=2$ $SO(5)$ discriminant [16]

$$s_2^2 (27\Lambda^{12} - \Lambda^6 s_1^3 - 36\Lambda^6 s_1 s_2 + s_1^4 s_2 + 8s_1^2 s_2^2 + 16s_2^3)^2 = 0, \tag{36}$$

where $s_1 = u_2$ and $s_2 = u_4 - u_2^2/2$ according to (26). Thus we see the discrepancy between (35) and (36) which implies that our simple assumption of $W_\Delta = 0$ does not work. Inspecting (35) and (36), however, we notice how to remedy the difficulty. Instead of (28) we take a tree-level superpotential

$$W = g_2 s_1 + g_4 s_2 = g_2 u_2 + g_4 \left(u_4 - \frac{1}{2} u_2^2 \right). \tag{37}$$

The classical vacuum condition is

$$W'(\Phi) = (g_2 - g_4 u_2)\Phi + g_4 \Phi^3 = 0. \tag{38}$$

To proceed, therefore, we can make use of the results obtained in the foregoing analysis just by making the replacement

$$\begin{aligned} g_4 &\rightarrow \tilde{g}_4 = g_4, \\ g_2 &\rightarrow \tilde{g}_2 = g_2 - u_2 g_4. \end{aligned} \tag{39}$$

(especially evaluation of $M_{\rm ad}$ is not invalidated because $\mathrm{Tr}\,\delta\Phi = 0$.) The eigenvalues of Φ are now determined in a self-consistent manner by

$$W'(x) = \tilde{g}_2 x + \tilde{g}_4 x^3 = \tilde{g}_4 x \left(x^2 + \frac{\tilde{g}_2}{\tilde{g}_4} \right) = \tilde{g}_4 x (x^2 - a_1^2) = 0. \tag{40}$$

Then we have $u_2^{cl} = 2a_1^2 = -2\tilde{g}_2/\tilde{g}_4$ and $\tilde{g}_2 = -g_2$ from (39), which leads to

$$a_1^2 = \frac{g_2}{g_4}. \tag{41}$$

Substituting this in (33) we calculate $\langle s_i \rangle$ and find the relation of s_i which is precisely the discriminant (36) except for the classical singularity at $\langle s_2 \rangle = 0$.

The above $SO(5)$ result indicates that an appropriate mixing term with respect to u_{2i} variables in a microscopic superpotential will be required for $SO(2N_c+1)$ theories. We are led to assume

$$W = \sum_{i=1}^{N_c-1} g_{2i} u_{2i} + g_{2N_c} s_{N_c} \tag{42}$$

for the gauge group $SO(2N_c+1)$ with $N_c \geq 3$. Then the following analysis is analogous to the $SO(5)$ theory. First of all notice that

$$s_{N_c} = u_{2N_c} - u_{2(N_c-1)} u_2 + (\text{polynomials of } u_{2k},\ 1 \leq k < N_c - 1). \tag{43}$$

Therefore the eigenvalues of Φ are given by the roots of (29) with the replacement

$$\begin{aligned} g_{2N_c} &\rightarrow \tilde{g}_{2N_c} = g_{2N_c}, \\ g_{2(N_c-1)} &\rightarrow \tilde{g}_{2(N_c-1)} = g_{2(N_c-1)} - u_2 g_{2N_c}. \end{aligned} \tag{44}$$

Then we have $u_2 = a_1^2 + \sum_{k=1}^{N_c-1} a_k^2 = a_1^2 - \tilde{g}_{2(N_c-1)}/\tilde{g}_{2N_c}$ and find

$$a_1^2 = \frac{g_{2(N_c-1)}}{g_{2N_c}}. \tag{45}$$

It follows that the effective superpotential is given by

$$W_L = W_L^{cl} \pm 2\sqrt{g_{2N_c} g_{2(N_c-1)}} \Lambda^{2N_c-1}. \tag{46}$$

The vacuum expectation values of gauge invariants are obtained from W_L as

$$\begin{aligned} \langle s_n \rangle &= s_n^{cl}(g), \qquad 1 \le n \le N_c - 2 \\ \langle s_{N_c-1} \rangle &= s_{N_c-1}^{cl}(g) \pm \frac{1}{a_1} \Lambda^{2N_c-1}, \\ \langle s_{N_c} \rangle &= s_{N_c}^{cl}(g) \pm a_1 \Lambda^{2N_c-1}. \end{aligned} \tag{47}$$

For these $\langle s_i \rangle$ the curve describing the $N = 2$ $SO(2N_c+1)$ theory [16] is shown to be degenerate as follows:

$$\begin{aligned} y^2 &= \langle \det(x - \Phi) \rangle^2 - 4x^2 \Lambda^{2(2N_c-1)} \\ &= (x^{2N_c} - \langle s_1 \rangle x^{2(N_c-1)} - \cdots - \langle s_{N_c-1} \rangle x^2 - \langle s_{N_c} \rangle + 2x\Lambda^{2N_c-1}) \\ &\quad \times (x^{2N_c} - \langle s_1 \rangle x^{2(N_c-1)} - \cdots - \langle s_{N_c-1} \rangle x^2 - \langle s_{N_c} \rangle - 2x\Lambda^{2N_c-1}) \\ &= \left\{ (x^2 - a_1^2)^2 (x^2 - a_2^2) \cdots (x^2 - a_{N_c-1}^2) \pm \Lambda^{2N_c-1} \left(-\frac{x^2}{a_1} - a_1 + 2x \right) \right\} \\ &\quad \times \left\{ (x^2 - a_1^2)^2 (x^2 - a_2^2) \cdots (x^2 - a_{N_c-1}^2) \pm \Lambda^{2N_c-1} \left(-\frac{x^2}{a_1} - a_1 - 2x \right) \right\} \\ &= (x^2 - a_1^2)^2 \left((x + a_1)^2 (x^2 - a_2^2) \cdots (x^2 - a_{N_c-1}^2) \mp \frac{\Lambda^{2N_c-1}}{a_1} \right) \\ &\quad \times \left((x - a_1)^2 (x^2 - a_2^2) \cdots (x^2 - a_{N_c-1}^2) \mp \frac{\Lambda^{2N_c-1}}{a_1} \right). \end{aligned} \tag{48}$$

Thus we see the theory with the superpotential (42) recover the $N = 2$ curve correctly with the assumption $W_\Delta = 0$. As in the $SO(2N_c)$ case, the singularity at $\langle s_{N_c} \rangle = 0$, which corresponds to the classical $SO(3) \times U(1)^{N_c-1}$ vacuum, does not arise in our theory.

We remark that the particular form of superpotential (42) is not unique to derive the singularity manifold. In fact we may start with a superpotential

$$W = \sum_{i=1}^{N_c-1} g_{2i} \left(u_{2i} + h_i(s) \right) + g_{2N_c} \left(s_{N_c} + h_{N_c}(s) \right), \tag{49}$$

where $h_i(s)$ are arbitrary polynomials of s_j with $j \geq N_c - 2$, to verify the $N = 2$ curve. However, we are not allowed to take a superpotential such as $W = \sum_{i=1}^{N_c} g_{2i} s_i$, because there are no $SU(2) \times U(1)^{N_c-1}$ vacua (there exist no solutions for $\tilde{g}_{2(N_c-1)}$). Note also that there are no $SO(3) \times U(1)^{N_c-1}$ vacua in the theory with superpotential (49).

Finally we discuss the $Sp(2N_c)$ gauge theory. The adjoint superfield Φ is a $2N_c \times 2N_c$ tensor which is subject to

$$ {}^t\Phi = J\Phi J \quad \Longleftrightarrow \quad J\Phi \text{ is symmetric}, \tag{50} $$

where $J = \mathrm{diag}(i\sigma_2, \cdots, i\sigma_2)$. Let us assume a tree-level superpotential

$$ W = \sum_{n=1}^{N_c} g_{2n} u_{2n}, \qquad u_{2n} = \frac{1}{2n} \mathrm{Tr}\, \Phi^{2n}. \tag{51} $$

Then our analysis will become quite similar to that for $SO(2N_c+1)$. The classical vacuum with unbroken $SU(2) \times U(1)^{N_c-1}$ gauge group corresponds to

$$ J\Phi = \mathrm{diag}(\sigma_1 a_1,\ \sigma_1 a_1,\ \sigma_1 a_2, \cdots, \sigma_1 a_{N_c-1}), \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{52} $$

The scale matching relation becomes

$$ \Lambda^{2\cdot(N_c+1)} = {\Lambda_L}^{3\cdot 2\cdot \frac{1}{2}} a_1^4 \left(\prod_{i=2}^{N_c-1} (a_1^2 - a_i^2) \right)^2 (M_{\mathrm{ad}})^{-1}. \tag{53} $$

Since the $SU(2)$ adjoint mass is given by $M_{\mathrm{ad}} = g_{2N_c} a_1^2 \prod_{i=2}^{N_c-1} (a_1^2 - a_i^2)$ we get ${\Lambda_L}^3 = g_{2N_c} \Lambda^{2(N_c+1)} / a_1^2$. The low-energy effective superpotential thus turns out to be

$$ W_L = W_{cl} + 2 \frac{g_{2N_c}}{a_1^2} \Lambda^{2(N_c+1)}. \tag{54} $$

Checking the result with $Sp(4)$ we encounter the same problem as in the $SO(5)$ theory. Instead of (51), thus, we take a superpotential in the form (37), reproducing the $N = 2$ $Sp(4)$ curve [17]. Similarly, for $Sp(2N_c)$ we study a superpotential (42). It turns out that $\langle s_i \rangle$ are calculated as

$$ \begin{aligned} \langle s_n \rangle &= s_n^{cl}(g), \qquad 1 \leq n \leq N_c - 2, \\ \langle s_{N_c-1} \rangle &= s_{N_c-1}^{cl}(g) - \frac{2}{a_1^4} \Lambda^{2(N_c+1)}, \\ \langle s_{N_c} \rangle &= s_{N_c}^{cl}(g) + \frac{4}{a_1^2} \Lambda^{2(N_c+1)}. \end{aligned} \tag{55} $$

These satisfy the $N=2$ $Sp(2N_c)$ singularity condition [17] since the curve exhibits the quadratic degeneracy

$$
\begin{aligned}
x^2y^2 &= \Big(x^2\left\langle \det(x-\Phi)\right\rangle + \Lambda^{2(N_c+1)}\Big)^2 - \Lambda^{4(N_c+1)} \\
&= (x^{2(N_c+1)} - \langle s_1\rangle x^{2N_c} - \cdots - \langle s_{N_c-1}\rangle x^4 - \langle s_{N_c}\rangle x^2 + 2\Lambda^{2(N_c+1)}) \\
&\quad \times (x^{2(N_c+1)} - \langle s_1\rangle x^{2N_c} - \cdots - \langle s_{N_c-1}\rangle x^4 - \langle s_{N_c}\rangle x^2) \\
&= \left\{ x^2(x^2-a_1^2)^2(x^2-a_2^2)\cdots(x^2-a_{N_c-1}^2) + 2\Lambda^{2(N_c+1)}\left(\left(\frac{x}{a_1}\right)^4 - 2\left(\frac{x}{a_1}\right)^2 + 1\right)\right\} \\
&\quad \times \Big(x^2 \det(x-\Phi_{cl})\Big) \\
&= (x^2-a_1^2)^2\left(x^2(x^2-a_2^2)\cdots(x^2-a_{N_c-1}^2) + \frac{\Lambda^{2(N_c+1)}}{a_1^4}\right) \times \Big(x^2\det(x-\Phi_{cl})\Big). \qquad (56)
\end{aligned}
$$

It should be mentioned that our remarks on $SO(2N_c+1)$ theories also apply here.

Now that we have found the $N=1$ effective superpotentials which can be used to derive the $N=2$ curves we wish to discuss $N=1$ SCFT. Recently large classes of novel $N=2$ SCFT have been shown to exist by fine-tuning the moduli coordinates at the points of coexisting mutually non-local massless dyons in the $N=2$ Coulomb phase [6], [18], [19]. On the other hand, an advantage of the present “integrating in” approach lies in the fact that we can explicitly read off how the microscopic parameters in the $N=1$ theory are related to the $N=2$ moduli coordinates. Therefore the coupling constants in our effective superpotentials are easily adjusted to the values of $N=2$ non-trivial critical points. Upon doing so we expect new classes of $N=1$ SCFT to be realized. This was first exploited by Argyres and Douglas in the $N=2$ $SU(3)$ gauge theory [6]. In the following we shall show that a mass gap in the $N=1$ confining phase of $SU(N_c)$ and $SO(2N_c)$ theories vanishes when the $N=1$ parameters are tuned as described above. Thus non-trivial $N=1$ fixed points will be identified.

We first consider the $N=2$ $SU(N_c)$ theory with $N_c\geq 3$. It was shown that the $N=2$ highest critical points exist at $\langle u_i\rangle=0$ for $2\leq i\leq N_c-1$ and $\langle u_{N_c}\rangle = \pm 2\Lambda^{N_c}$ [19]. The critical points are featured by Z_{N_c} symmetry. When we approach these points under the $N=1$ perturbation it is clear from (2) and (12) that the coupling constants in (1) become

$$
g_i \longrightarrow 0, \qquad 2\leq i\leq N_c-1 \qquad (57)
$$

and the superpotential reduces to

$$W_{\rm crit} = \frac{g_{N_c}}{N_c} \,{\rm Tr}\, \Phi^{N_c}. \tag{58}$$

In our $N = 1$ theory there exists a mass gap due to dyon condensation and the gauge field gets a mass by the magnetic Higgs mechanism. Let us check how the gap behaves in the limit (57). For this purpose it is convenient to consider a macroscopic $N = 1$ superpotential W_m obtained from the effective low-energy $N = 2$ Abelian theory. We denote by A_i the $N = 1$ chiral superfield of (N_c-1) $N = 2$ $U(1)$ multiplets and by $M_i, \tilde{M}_i$ the $N = 1$ chiral superfields of $N = 2$ dyon hypermultiplets. The superpotential W_m then takes the form [2], [6]

$$W_m = \sqrt{2} \sum_{i=1}^{N_c-1} A_i M_i \tilde{M}_i + \sum_{i=2}^{N_c} g_i U_i, \tag{59}$$

where U_i represent the superfields corresponding to ${\rm Tr}\, \Phi^i$ and their lowest components have the expectation values $\langle u_i \rangle$. The equation of motion is given by

$$-\frac{g_n}{\sqrt{2}} = \sum_{i=1}^{N_c-1} \frac{\partial a_i}{\partial \langle u_n \rangle} m_i \tilde{m}_i, \qquad 2 \leq n \leq N_c \tag{60}$$

and

$$a_i m_i = a_i \tilde{m}_i = 0, \qquad 1 \leq i \leq N_c - 1 \tag{61}$$

where a_i, m_i and $\tilde{m}_i$ stand for the expectation values of the lowest components of A_i, M_i and $\tilde{M}_i$ respectively. The D-flatness condition implies $|m_i| = |\tilde{m}_i|$.

Concentrate now on the singular point where we have only one massless dyon, say $M_1, \tilde{M}_1$. Note that this is the $N = 1$ vacuum for which we have derived the effective superpotential (8). Then, $a_1 = 0$ and $a_i \neq 0$ for $i \neq 1$, and hence (61) yields $m_i = 0$ for $i \neq 1$. Eq.(60) is rewritten as

$$\frac{g_i}{g_{N_c}} = \frac{\partial a_1 / \partial \langle u_i \rangle}{\partial a_1 / \partial \langle u_{N_c} \rangle}, \qquad 2 \leq i \leq N_c - 1. \tag{62}$$

We bring the system to the Z_{N_c} critical points by tuning a parameter ϵ

$$\langle u_n \rangle \pm 2\Lambda^{N_c} \delta_{n,N_c} = c_n \epsilon^n, \qquad c_n = {\rm const.} \tag{63}$$

where ϵ is an overall mass scale. Following [6], [19] we evaluate

$$\frac{\partial a_1}{\partial \langle u_i \rangle} \simeq \epsilon^{N_c/2+1-i}, \qquad 2 \leq i \leq N_c. \tag{64}$$

Thus we find

$$\frac{g_i}{g_{N_c}} \simeq \epsilon^{N_c-i} \longrightarrow 0 \tag{65}$$

for $2 \leq i \leq N_c - 1$. This agrees with (57).

The gap in the $U(1)$ factor arises from the dyon condensation m_1. From (60) we obtain the scaling behavior

$$m_1 = \Big(- \frac{g_{N_c}}{\sqrt{2}\partial a_1/\partial\langle u_{N_c}\rangle} \Big)^{1/2} \simeq \sqrt{g_{N_c}}\, \epsilon^{(N_c-2)/4} \longrightarrow 0. \tag{66}$$

Thus the gap vanishes as we approach the Z_{N_c} critical point. Therefore the Z_{N_c} vacua of our $N=1$ theory characterized by a superpotential (58) is a non-trivial fixed point.

We now turn to the $SO(2N_c)$ theory with $N_c \geq 3$. The $N=2$ $SO(2N_c)$ theory possesses the highest critical points at $\langle u_{2i}\rangle = 0$ $(1 \leq i \leq N_c - 2)$, $\langle v\rangle = 0$ and $\langle u_{2(N_c-1)}\rangle = \pm 2\Lambda^{2(N_c-1)}$ [19]. In the $N=1$ superpotential (13) this criticality corresponds to

$$g_{2i} \longrightarrow 0, \qquad \lambda \longrightarrow 0 \tag{67}$$

for $1 \leq i \leq N_c - 2$ and we have

$$W_{\rm crit} = \frac{g_{2(N_c-1)}}{2(N_c-1)} \,{\rm Tr}\, \Phi^{2(N_c-1)}. \tag{68}$$

Let us show that an $N=1$ gap vanishes in this limit by looking at the singular point where a single massless dyon exists. A similar analysis to the $SU(N_c)$ theory gives us the vacuum condition

$$\frac{g_{2i}}{g_{2(N_c-1)}} = \frac{\partial a_1/\partial\langle u_{2i}\rangle}{\partial a_1/\partial\langle u_{2(N_c-1)}\rangle}, \qquad \frac{\lambda}{g_{2(N_c-1)}} = \frac{\partial a_1/\partial\langle v\rangle}{\partial a_1/\partial\langle u_{2(N_c-1)}\rangle} \tag{69}$$

for $1 \leq i \leq N_c - 2$. The critical limit is taken through the parametrization

$$\begin{aligned} &\langle u_{2n}\rangle \pm 2\Lambda^{2(N_c-1)}\delta_{n,N_c-1} = c_n \epsilon^{2n}, \qquad 1 \leq n \leq N_c - 1, \\ &\langle v\rangle = c\,\epsilon^{N_c}, \end{aligned} \tag{70}$$

where ϵ is an overall mass scale and c_n, c are ϵ-independent constants. We obtain from (69) that

$$\begin{aligned} &\frac{g_{2i}}{g_{2(N_c-1)}} \simeq \epsilon^{2(N_c-1-i)} \longrightarrow 0, \qquad 1 \leq i \leq N_c - 2, \\ &\frac{\lambda}{g_{2(N_c-1)}} \simeq \epsilon^{N_c-2} \longrightarrow 0 \end{aligned} \tag{71}$$

in agreement with (67). The gap in the $U(1)$ factor scales as

$$m_1 = \Big(- \frac{g_{2(N_c-1)}}{\sqrt{2}\partial a_1/\partial\langle u_{2(N_c-1)}\rangle}\Big)^{1/2} \simeq \sqrt{g_{2(N_c-1)}}\,\epsilon^{(N_c-2)/2} \longrightarrow 0. \tag{72}$$

Thus our $N=1$ $SO(2N_c)$ theory with a superpotential (68) has a non-trivial fixed point.

In conclusion we have shown how to derive the curves for the Coulomb phase of $N=2$ Yang-Mills theories with classical gauge groups by means of the $N=1$ confining phase superpotential. Transferring the critical points in the $N=2$ Coulomb phase to the $N=1$ theories we have found non-trivial $N=1$ SCFT with the adjoint matter governed by a superpotential. This SCFT certainly has a connection with the non-Abelian Coulomb phase of the Kutasov-Schwimmer model [20], [21], [12]. To further explore this connection it will be interesting to investigate theories containing the additional fundamental matter multiplets.

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